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SINTESI DELLA TESI

Enriques-Kodaira classification of Complex Algebraic Surfaces

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Our principal aim is to describe and prove the classification of complex algebraic surfaces, a result due to the Italian school of Algebraic Geometry at the beginning of the XX century, and in particular to Enriques. This classification is done up to birational maps, but it is different from the classification of curves. In fact, in the case of curves there is a unique nonsingular projective model for each equivalence class, whereas in the case of surfaces this model is not unique. Therefore we will need to deal with something different: the minimal models. In fact, apart from the case of ruled surfaces (i.e. those birational to the product of a curve and the projective line), these are unique and allow us to obtain birationally equivalent surfaces by successive blow-ups.

Thus the classification problem will split in two parts: ruled surfaces, which require special considerations, and non-ruled ones, for which it will suffice to classify minimal models. For this, we will need some birational invariants, which capture the geometric peculiarities of each class, as we will see. A first rough classification is achieved by means of Kodaira dimension, κ, which is defined as the largest dimension of the image of the surface in a projective space by the rational map determined by the linear system $|nK|$, or as -1 if $|nK| = \varnothing$ for every n. This will allow us to identify four big classes of surfaces.

A more precise classification within the above classes is achieved through the invariants: $q(S) := h^1(S, \mathcal{O}_S)$, $p_g(S) := h^2(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S(K))$ and, for $n \in \mathbb{N}$, $P_n(S) := h^0(S, \mathcal{O}_S(nK))$.

In the first chapter we collect heterogeneous results which will be used to achieve the classification. In particular, we study curves as divisors on surfaces, and their intersection numbers. This will be one of the most useful tools to prove the important and deep theorems that follow.

The second chapter treats ruled surfaces, which will constitute a whole class of surfaces (those with $\kappa = -1$). In the third we prove the fundamental Castelnuovo's theorem, that is a criterion for a surface to be rational. From this, other important theorems will follow.

In the fourth chapter we prove the other fundamental Enriques theorem, a criterion for a surface to be ruled. We also study in detail surfaces with $p_g = 0$ and $q \geq 1$. This work will turn out to be useful also in the last chapter, where we will complete the classification of surfaces according to their Kodaira dimension. In particular, we will provide a more detailed description of the case $\kappa = 0$, and collect some basic facts on K3 surfaces.

We will always work on the complex field $\mathbb{C}$. Serre proved in his paper Géometrie Algébrique et Géométrie Analytique ([GAGA]) that over the complex field, the algebraic point of view and the analytical point of view are essentially equivalent: without getting into details, we can at least say that subvarieties correspond to submanifolds, morphisms correspond to holomorphic maps and rational maps to meromorphic maps. We will sometimes use this deep relation to ease some proofs.

Our first task is to deepen some aspects of the theory of curves, since they turn out to be a fundamental instrument to study surfaces. We start with some results on abstract curves, and then focus on some properties of curves on surfaces. The main interest of these lies in the fact that divisors of a surface are curves. We will in particular introduce the intersection numbers, that somehow describe the intersection of two curves (and a lot more), and that will play an expecially crucial role in the study of surfaces.

Definition 0.1. Let C, C' be two *distinct, irreducible* curves on a surface S, $x \in C \cap C'$, $\mathcal{O}_x$ the local ring of S at x. Let f, g be local equations at x for C, C' respectively. We then define the *intersection multiplicity* of C and C' at x as

$$(C.C')_x := dim_{\mathbb{C}}\ (\mathcal{O}_x/(f,g))$$

and we define the *intersection number* of C and C' as

$$(C.C') := \sum_{x \in C \cap C'} (C.C')_x.$$

Definition 0.2. For $L, L' \in Pic(S)$, we define

$$(L.L') := \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}).$$

Theorem 0.0.1. *$(\ .\ )$ is a symmetric bilinear form on $Pic(S)$, such that*

$$\mathcal{O}_S(C).\mathcal{O}_S(C') = (C.C')$$

for every C, C' distinct irreducible curves on S.

The previous result allows us to define the intersection number of any two divisors.

Definition 0.3. If D, D' are two divisors on a surface S, we define their intersection number as:

$$D.D' := (\mathcal{O}_S(D).\mathcal{O}_S(D')).$$

By the theorem we have just stated, we can calculate this product by replacing D, or D', by a linearly equivalent divisor. This fact is very important for most of the applications of intersection numbers. As a consequence, it makes perfect sense to consider the *self-intersection number* of a divisor D, i.e. $D.D$, which we will often write D^2.

By means of intersection numbers, we have the two following fundamental results:

Theorem 0.0.2 (Riemann-Roch Theorem for surfaces)**.** *For all $D \in Div(S)$,*

$$\chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - D.K_S).$$

Theorem 0.0.3 (Genus formula)**.** *Let C be a nonsingular, irreducible curve on a surface S. Then the genus of C is given by*

$$g(C) = \frac{1}{2}(C^2 + C.K_S) + 1.$$

We then study birational maps between surfaces. We see that in this case they behave surprisingly well, since they can be completely described in terms of blow-ups.

Theorem 0.0.4 (Resolution of indeterminacies). *Let S be a nonsingular projective surface and $f : S \to \mathbb{P}^n$ a rational map. Then there exists a chain of blow-ups $\sigma_i : S_i \to S_{i-1}$, $i = 1, ..., m$, with $S_0 = S$, such that $f \circ \sigma_1 \circ ... \circ \sigma_m : S_m \to \mathbb{P}^n$ is regular.*

Theorem 0.0.5 (Structure of birational morphisms). *Let $f : S' \to S$ be a birational morphism between nonsingular projective surfaces. Then there exists a chain of surfaces and blowups $\sigma_i : S_i \to S_{i-1}$ for $i = 1, ..., k$, such that $S = S_0$, there is an isomorphism $\phi : S_k \xrightarrow{\cong} S'$, and $f = \sigma_1 \circ ... \circ \sigma_k \circ \phi^{-1}$.*

Theorem 0.0.6 (Structure of birational maps). *Let $f : S' \to S$ be a birational map between nonsingular projective surfaces. Then f can be thought of as a composition of a chain of blowups and a chain of blowdowns; more precisely, there exist two chains of blowups, $\sigma_i : S'_i \to S'_{i-1}$, $i = 1, ..., m$, $S'_0 = S'$, $\tau_j : S_j \to S_{j-1}$, $j = 1, ..., l$, $S_0 = S$, and a surface $Z = S'_m = S_l$, such that $f \circ \sigma_1 \circ ... \circ \sigma_m = \tau_1 \circ ... \circ \tau_l$.*

The following theorem guarantees that it is possible the resolve the singularities of a curve by successive blowups.

Theorem 0.0.7. *Let C be an irreducible curve on a nonsingular surface S. Then there exist a surface S' and a regular map $f : S' \to S$ such that f is a composite of blowups $S' \to S_1 \to ... \to S_n \to S$ and the birational transform $\tilde{C}$ of C on S' is nonsingular.*

With the same notation of the previous theorem, if p is a singular point of C, all the singular points that arise out of p by blowing up are called *infinitely near* to p. We have:

Proposition 0.0.8. *Let C be an irreducible curve on a nonsingular surface S, and let $\overline{C}$ be its desingularization. Let $\{p_i\}_i$ be the set of all the infinitely near points, and let k_i be the multiplicity of such p_i. Then*

$$g(\overline{C}) = 1 + \frac{1}{2}(C^2 + C.K_S) - \sum_i \frac{k_i(k_i - 1)}{2}.$$

Corollary 0.0.9. *If $S = \mathbb{P}^2$ and C is a curve of degree n, we have*

$$g(\overline{C}) = \frac{(n-1)(n-2)}{2} - \sum_i \frac{k_i(k_i-1)}{2}.$$

Corollary 0.0.10. *We always have $C^2 + C.K_S = 2g(\overline{C}) - 2 \geq -2$.*

In particular, $C^2 + C.K_S = -2 \iff C$ is nonsingular and $g(\overline{C}) = g(C) = 0 \iff C \simeq \mathbb{P}^1$.

We know that the exceptional curve of a blowup is isomorphic to $\mathbb{P}^1$ and has self-intersection number -1. We will see now that the converse is also true, i.e. a curve with these properties is the exceptional curve of some blowup. Then it can be contracted without affecting the nonsingularity of the surface, hence the name of this deep result: Castelnuovo's contractibility criterion.

Theorem 0.0.11 (Castelnuovo's contractibility criterion)**.** *Let S be a surface and $E \subset S$ a curve isomorphic to $\mathbb{P}^1$ with $E^2 = -1$. Then E is an exceptional curve on S, i.e. the exceptional curve of a blow-up $\sigma : S \to S'$ at a point of S', where S' is a smooth surface.*

Next we deduce an exact sequence

$$0 \to T \to Pic(S) \xrightarrow{c} NS(S) \to 0$$

where $NS(S) \subset H^2(S, \mathbb{Z})$ is a finitely generated group, called the Néron-Severi group of S. We then describe why they are useful.

If $f : S \to S'$ is a birational morphism between surfaces, we know that it can be written as composition of, say, n, blowups, and therefore the Néron-Severi groups are related by an isomorphism $NS(S) \simeq NS(S') \oplus \mathbb{Z}^n$. Now their importance lies in the fact that they are finitely generated: thus we have

$$n = rg\ NS(S) - rg\ NS(S')$$

which means that n is independent of the factorization (it depends only on S and S').

In particular, we also have that if $S' = S$, f must be an isomorphism.

Now let S be a surface, and $B(S)$ the set of nonsingular, projective surfaces birational to S. We can introduce a partial order relationship on $B(S)$, namely $S_1 \leq S_2$ if there exists a birational morphism $S_2 \to S_1$, and we say that S_2 *dominates* S_1. If S' is a minimal element in $B(S)$, we say that S is a *minimal surface*. It is easy to see that:

Proposition 0.0.12. *Every surface dominates a minimal surface.*

Thus for every surface S, $B(S)$ contains at least one minimal surface, and all other surfaces arise from blowups of the minimal ones. Minimal surfaces play therefore a crucial role in the classification, inasmuch as we will see that every nonruled surface has a unique minimal model.

Another important tool to classify surfaces is the Kodaira dimension, a birational invariant through which we will divide the set of all surfaces into four classes.

Recall that if V a smooth projective variety and K is a canonical divisor of V, the *Kodaira dimension* $\kappa(V)$ of V is defined as

$$\kappa(V) := max_{n\geq 1}\{dim\ \phi_{nK}(V)\}$$

where we put $dim\ \phi_{nK}(V) := -1$ if $|nK| = \varnothing$.

Obviously, $\kappa(V) \in \{-1, 0, 1, ..., dim\ V\}$.

We then consider ruled surfaces, which, as we will see, by Enriques theorem are exactly all surfaces with Kodaira dimension -1.

A *ruled surface* is a surface S birationally equivalent to $C \times \mathbb{P}^1$, where C is a nonsingular curve. In particular, every rational surface is ruled, since $\mathbb{P}^2 \approx \mathbb{P}^1 \times \mathbb{P}^1$.

A *geometrically ruled surface* over a non singular curve C is a surface S together with a smooth morphism $p : S \to C$ whose fibres are isomorphic to $\mathbb{P}^1$.

$C \times \mathbb{P}^1$ is both a ruled and a geometrically ruled surface (considering the canonical projection $C \times \mathbb{P}^1 \to C$).

Next consider a $\mathbb{P}^1-$bundle X over a smooth curve C, with structural morphism $p : X \to C$. Then there exists an open $U \subset C$ such that $p^{-1}(U)$ is isomorphic to $U \times \mathbb{P}^1$. Hence X is birational to $C \times \mathbb{P}^1$, i.e. is a ruled surface. But since every fibre of p is isomorphic to $\mathbb{P}^1$, X is also a geometrically ruled surface. This observation can be reversed in the following sense: every geometrically ruled surface is in fact a ruled surface, and also a $\mathbb{P}^1-$bundle. This is proved by the following deep result.

Theorem 0.0.13 (Noether-Enriques)**.** *Let S be a surface, C a smooth curve and $p : S \to C$ a morphism such that, for some point $x \in C$, p is smooth over x and $p^{-1}(x) \simeq \mathbb{P}^1$. Then there exists an open neighbourhood $U \subset C$ of x and an isomorphism $p^{-1}(U) \simeq U \times \mathbb{P}^1$ such that the following diagram is commutative*

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\simeq} & U \times \mathbb{P}^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ U & \xrightarrow[id]{} & U \end{array}$$

where π_1 is the canonical projection.

In particular, S is a ruled surface.

For any rank 2 vector bundle E over a smooth curve C, the associated projective bundle $\mathbb{P}_C(E)$ is a $\mathbb{P}^1-$bundle, hence a geometrically ruled surface. Next we prove that every geometrically ruled surface is of this type, and that geometrically ruled surfaces over irrational curves are the minimal models of ruled surfaces. Therefore, our main task will be to study in some detail rank 2 vector bundles over smooth curves.

Proposition 0.0.14. *Every geometrically ruled surface $p : S \to C$ is C-isomorphic to $\mathbb{P}_C(E)$ for some rank 2 vector bundle over C.*

Moreover, $\mathbb{P}_C(E)$ is C-isomorphic to $\mathbb{P}_C(E') \iff E' \simeq E \otimes L$ for some line bundle L on C.

Theorem 0.0.15. *Let C be a smooth* non rational *curve. The minimal models of $C \times \mathbb{P}^1$ are the geometrically ruled surfaces over C (that is, the projective bundles $\mathbb{P}_C(E)$).*

We deal then with the cases C rational and C elliptic.

Proposition 0.0.16. *(i) Every rank 2 vector bundle on $\mathbb{P}^1$ is decomposable.*

(ii) Every rank 2 vector bundle on an elliptic curve C satisfies one of the following properties:

(1) decomposable;

(2) isomorphic to $E \otimes L$, where $L \in Pic(C)$ and E is a non-trivial extension of $\mathcal{O}_C$ by $\mathcal{O}_C$;

(3) isomorphic to $E \otimes L$, where $L \in Pic(C)$ and E is a non-trivial extension of $\mathcal{O}_C(p)$ by $\mathcal{O}_C$.

Corollary 0.0.17. *Every geometrically ruled surface S over $\mathbb{P}^1$ is isomorphic to one of the following (called* Hirzebruch surfaces*):*

$$H_n := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \quad for \ n \geq 0.$$

After introducing the *tautological bundle* of S, written $\mathcal{O}_S(1)$, we prove the following result:

Proposition 0.0.18. *Let $S = \mathbb{P}_C(E)$ be a geometrically ruled surface with associated morphism $p : S \to C$, and let h be the divisor class of $\mathcal{O}_S(1)$ in $Pic(S)$ or $H^2(S, \mathbb{Z})$. As always, let $F \in Pic(S)$ be a generic fibre, and $f \in H^2(S, \mathbb{Z})$ its class. Then:*

(i) $Pic(S) = p^* Pic(C) \oplus \mathbb{Z}h$;

(ii) $H^2(S, \mathbb{Z}) = \mathbb{Z}h \oplus \mathbb{Z}f$;

(iii) $h^2 = deg\ E$;

(iv) $[K_S] = -2h + (deg\ E + 2g(C) - 2)f$ in $H^2(S, \mathbb{Z})$.

We can now calculate the numerical invariants of ruled surfaces.

Proposition 0.0.19. *If S is a ruled surface over C:*

$$q(S) = g(C), \quad p_g(S) = 0, \quad P_n(S) = 0 \ \forall \ n \geq 2.$$

If S is geometrically ruled over C:

$$K_S^2 = 8(1 - g(C)), \quad b_2(S) = 2.$$

Later on, we will be able to describe minimal rational surfaces, which are the already seen H_n for $n \neq 1$ (in fact we see in a moment that H_1 is the blow-up of the plane). For now we can prove that they are minimal for $n \neq 1$ and not isomorphic to one another.

Proposition 0.0.20. *(i) Pic $H_n = \mathbb{Z}h \oplus \mathbb{Z}f$; $f^2 = 0, f.h = 1, h^2 = n$.*

(ii) For every $n > 0$, H_n contains a unique irreducible curve B such that $B^2 < 0$. If b is its class in $H^2(H_n, \mathbb{Z})$, $b = h - nf$ and $b^2 = -n$.

(iii) H_n and H_m are isomorphic if and only if $n = m$. For very $n \neq 1$, H_n is minimal. H_1 is isomorphic to the blow-up of $\mathbb{P}^2$ at a point.

Then we prove a somehow technical result, which leads us to a few fundamental theorems.

Proposition 0.0.21. *If S is a minimal surface with $q = P_2 = 0$, then there exists a smooth rational curve C on S such that $C^2 \geq 0$.*

A first corollary is the following:

Theorem 0.0.22 (Castelnuovo's Rationality Criterion)**.** *If S is a surface with $q = P_2 = 0$, then S is rational.*

We already know that for every $n \neq 1$, H_n is minimal, and that a minimal model of H_1 is $\mathbb{P}^2$. Now we can prove that there are no other minimal rational surfaces.

Theorem 0.0.23. *If S is a minimal rational surface, then S is isomorphic to one of the following:*

(1) $\mathbb{P}^2$;

(2) H_n for some $n \neq 1$.

Finally, we can see that non-ruled minimal surfaces are unique up to isomorphisms.

Theorem 0.0.24. *Let S, S' be two non-ruled minimal surfaces. Then every birational map from one to the other is an isomorphism. Therefore, every non-ruled surface admits a unique minimal model (up to isomorphism).*

Enriques theorem is another fundamental result to achieve the classification, since it provides necessary and sufficient conditions for a surface to be ruled. It is a deep theorem which requires several preliminary results.

Theorem 0.0.25. *Let S be a minimal non-ruled surface with $p_g = 0,\ q \geq 1$. Then:*

(a) $q = 1,\ K^2 = 0$.

(b) $S \simeq (B \times F)/G$ with $B,\ F$ smooth irrational curves and G is a finite group acting on B and F.

(c) B/G is elliptic.

(d) F/G is rational.

(e) either B is elliptic (and G is a group of translations of B) or F is elliptic.

(f) G acts freely on $B \times F$ (i.e. $\pi : B \times F \to S$ is étale).

Conversely, let S be a surface with the properties $(b), ..., (f)$ above. Then S is minimal, non-ruled and has $p_g = 0,\ q = 1,\ K^2 = 0$.

To get to Enriques theorem from here, we need to study the behaviour of plurigenera of the surfaces we are dealing with.

Theorem 0.0.26. *Let $S \simeq (B \times F)/G$ be a minimal non-ruled surface with $p_g = 0, q \geq 1$ (in particular, either B or F is elliptic). Then:*

(1) either $P_4 \neq 0$ or $P_6 \neq 0$ (in particular, $P_{12} \neq 0$);

(2) if B and F are not both elliptic, there is an infinite increasing sequence of integers $\{n_i\}$ such that $\{P_{n_i}\}$ tends to infinity;

(3) if B and F are both elliptic, then $4K \sim 0$ or $6K \sim 0$ (in particular, $12K \sim 0$).

After these preliminaries we can prove without difficulties Enriques theorem.

Theorem 0.0.27 (Enriques' theorem)**.** *Let S be a surface with $P_4 = P_6 = 0$ (or equivalently $P_{12} = 0$). Then S is ruled.*

Corollary 0.0.28. *Let S be a surface. The following conditions are equivalent:*

(1) S is ruled;

(2) there is a non-exceptional *curve C on S such that $K.C < 0$;*

(3) for every divisor D, $|D + nK| = \varnothing$ for $n >> 0$;

(4) $P_n = 0\ \forall\ n$;

(5) $P_{12} = 0$.

Remark 1. It follows immediately from this corollary that

$$\kappa(S) = -1 \iff S \textit{ is ruled}$$

It is now useful to give the definition of bielliptic surfaces, since it follows from the previous results that these are the only surfaces with $p_g = 0,\ q \geq 1$ having $\kappa = 0$.

Definition 0.4. A *bielliptic surface* is a surface $S \simeq (E \times F)/G$, with E, F elliptic curves, G finite group of translations of E acting on F such that $F/G \simeq \mathbb{P}^1$.

We are now ready to describe the Enriques-Kodaira classification. By Enriques theorem, we have identified surfaces with $\kappa = -1$, which are exactly the ruled ones. Now we consider the other possibilities for Kodaira dimension, i.e. $\kappa \geq 0$. We know that a surface of each of these cases has a unique (up to isomorphism) minimal model. Therefore to achieve a birational classification it suffices to consider minimal surfaces.

To begin with, we consider surfaces with $\kappa = 1$. What we can say about them is that they all belong to the class of elliptic surfaces, i.e.

Definition 0.5. Let S be a surface and $p : S \to B$ a surjective morphism to a smooth curve B, whose generic fibre is an elliptic curve. Then S is called an *elliptic surface.*

Proposition 0.0.29. *Let S be a minimal surface with $\kappa = 1$. Then:*
(a) $K^2 = 0$;
(b) S is elliptic.

There exist plenty of elliptic surfaces with $\kappa \neq 1$, as the following result shows:

Proposition 0.0.30. *Let S be a minimal elliptic surface, whose elliptic fibration is $p : S \to B$. Write F_b for the fibre over $b \in B$. Then:*
(1) $K^2 = 0$.
(2) S is either ruled over an elliptic curve, or has $\kappa = 0$, or has $\kappa = 1$.
(3) If $\kappa = 1$, there exists an integer $d > 0$ such that

$$dK \sim \sum n_i F_{b_i} \qquad n_i \in \mathbb{N},\ b_i \in B$$

For r large, the system $|rdK|$ is base points free and the morphism to $\mathbb{P}^N$ it defines factors as $S \xrightarrow{p} B \xrightarrow{j} \mathbb{P}^N$, where j is an embedding.

Surfaces with Kodaira dimension 2 are called *of general type*; we can easily characterize a surface to be of general type.

Proposition 0.0.31. *Let S be a minimal surface. Then the following conditions are equivalent:*
(1) $\kappa(S) = 2$;
(2) $K^2 > 0$ and S is irrational;
(3) there exists an integer n_0 such that ϕ_{nK} from S to its image is birational for every $n \geq n_0$.

Apart from this result, these are the most difficult surfaces to classify further, since, as the name suggests, this family contains many surfaces very different from each other.

Now we focus on surfaces with $\kappa = 0$. We can provide a more precise classification according to the invariants q and p_g.

Theorem 0.0.32. *Let S be a minimal surface with $\kappa = 0$. Then S belongs to exactly one of the following 4 cases:*

(1) $p_g = 0,\ q = 0$. Then $2K \sim 0$, and S is called an Enriques surface*.*

(2) $p_g = 0,\ q = 1$. Then S is a bielliptic surface.

(3) $p_g = 1,\ q = 0$. Then $K \sim 0$, and S is called a K3 surface*.*

(4) $p_g = 1,\ q = 2$. Then $K \sim 0$, and S is called an Abelian surface*.*

Corollary 0.0.33. *Let S be a minimal surface with $\kappa(S) = 0$; then $4K \sim 0$ or $6K \sim 0$.*

In the end, the classification of Complex Algebraic Surfaces can be summarized as follows:

Theorem 0.0.34 (Enriques-Kodaira classification theorem)**.**

Let S be a nonsingular projective surface.

- *$\kappa(S) = -1 \iff S$ is ruled, and in this case $q(S) = 0 \iff S$ is rational.*

If $\kappa(S) \geq 0$, consider the minimal model S' of S (unique up to isomorphism).

- *If $\kappa(S') = 0$. Then $K^2 = 0$, and:*
  - *if $p_g = 0$ and $q = 0$, then S' is an Enriques surface;*
  - *if $p_g = 0$ and $q = 1$, then S' is a bielliptic surface;*
  - *if $p_g = 1$ and $q = 0$, then S' is a K3 surface;*
  - *if $p_g = 1$ and $q = 2$, then S' is an Abelian surface.*
- *If $\kappa(S') = 1$, S' is an elliptic surface; $K^2 = 0$.*
- *If $\kappa(S') = 2$, S' is a surface of general type; $K^2 > 0$.*